

4.6 Dirichlet's Hyperbolic Method

The result

$$\sum_{n \leq x} \frac{1}{n} = \log x + O(1) \quad (1)$$

was used to prove

$$\sum_{n \leq x} d(n) = x \log x + O(x), \quad (2)$$

where $d(n)$ is the divisor function.

In Chapter 2 the result (1) was improved to

Theorem 1 *There exists a constant $\gamma > 0$ such that*

$$\sum_{n \leq x} \frac{1}{n} = \log x + \gamma + O\left(\frac{1}{x}\right).$$

If this is used in the proof of (2) to give extra main terms and a smaller error, it fails. A method designed by Dirichlet to improve (2) can be given generally as

Theorem 2 *Dirichlet's Hyperbolic Method* *Assume that f and g are arithmetic functions and write*

$$F(x) = \sum_{1 \leq n \leq x} f(n) \quad \text{and} \quad G(x) = \sum_{1 \leq n \leq x} g(n).$$

Then

$$\sum_{1 \leq n \leq x} f * g(n) = \sum_{1 \leq a \leq U} f(a) G\left(\frac{x}{a}\right) + \sum_{1 \leq b \leq V} F\left(\frac{x}{b}\right) g(b) - F(U) G(V)$$

for any $UV = x$.

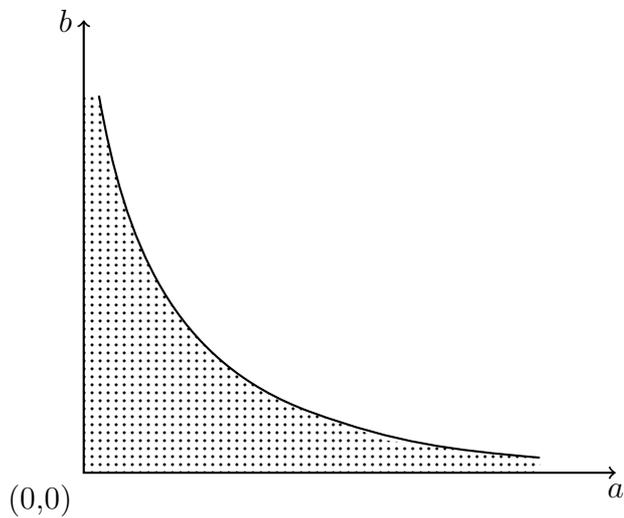
Proof From the definition of convolution we have

$$\sum_{1 \leq n \leq x} f * g(n) = \sum_{1 \leq n \leq x} \sum_{ab=n} f(a) g(b) = \sum_{1 \leq ab \leq x} f(a) g(b),$$

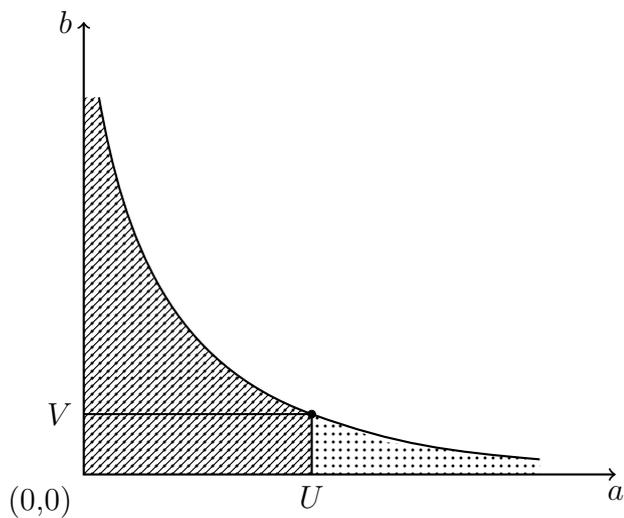
where the sum is over the set

$$S = \{(a, b) \in \mathbb{Z}^2 : ab \leq x\}.$$

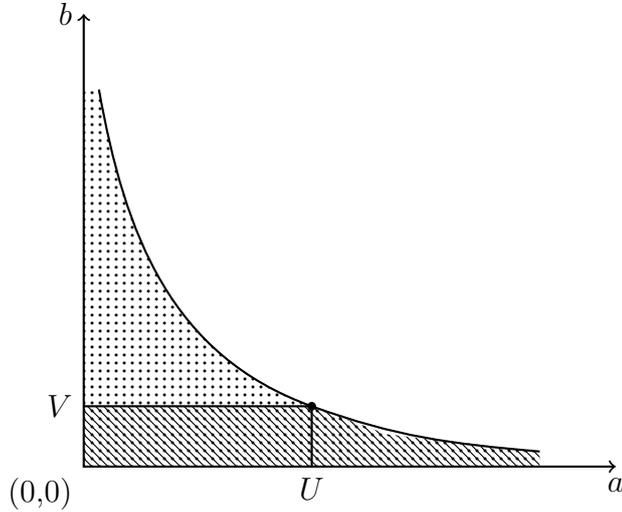
This is the set of ordered integer pairs under the hyperbola $ab = x$, giving the name to this method.



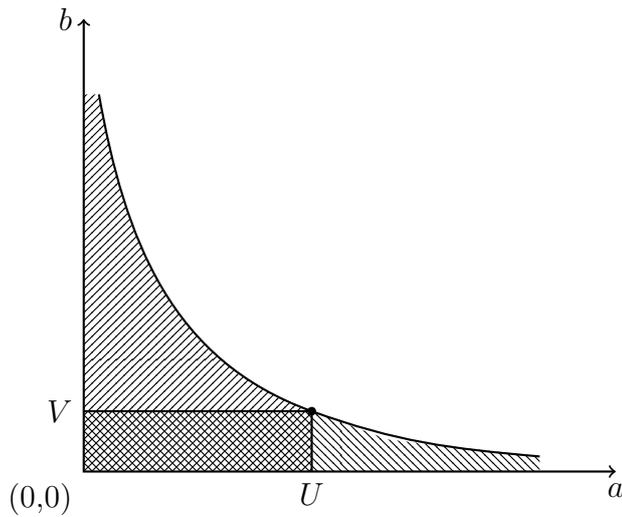
Now choose $U, V \geq 1$ such that $UV = x$. We first sum over the integer pairs (a, b) under the hyperbola with $a \leq U$.



Then we sum over the pairs with $b \leq V$.



But we will then have summed twice over the points in the rectangle that is the intersection of these two regions.



Thus we have to remove one of these summations over the points (a, b) with $a \leq U$ and $b \leq V$. In this way we obtain

$$\sum_{(a,b) \in S} f(a) g(b) = \sum_{ab \leq x} \sum_{a \leq U} f(a) g(b) + \sum_{ab \leq x} \sum_{b \leq V} f(a) g(b) \quad (3)$$

$$- \sum_{a \leq U} \sum_{b \leq V} f(a) g(b)$$

Alternative derivation of (3). Since $UV = x$ then, if (a, b) satisfies $ab \leq x$ it may be that $a \leq U$. But if this is **not** the case, i.e. $a \geq U$ then

$$b \leq x/a \leq x/U = V.$$

Thus either $a \leq U$ or $b \leq V$. Hence $S = S_1 \cup S_2$ where

$$S_1 = \{(a, b) \in \mathbb{Z}^2 : ab \leq x, a \leq U\} \quad \text{and} \quad S_2 = \{(a, b) \in \mathbb{Z}^2 : ab \leq x, b \leq V\}.$$

Therefore

$$\begin{aligned} \sum_{(a,b) \in S} f(a)g(b) &= \sum_{(a,b) \in S_1 \cup S_2} f(a)g(b) \\ &= \sum_{(a,b) \in S_1} f(a)g(b) + \sum_{(a,b) \in S_2} f(a)g(b) - \sum_{(a,b) \in S_1 \cap S_2} f(a)g(b), \end{aligned}$$

since to sum over the elements of S_1 and then over the elements of S_2 means that you will have summed over the elements of $S_1 \cap S_2$ twice. Yet

$$\begin{aligned} S_1 \cap S_2 &= \{(a, b) \in \mathbb{Z}^2 : ab \leq x, a \leq U, b \leq V\} \\ &= \{(a, b) \in \mathbb{Z}^2 : a \leq U, b \leq V\}, \end{aligned}$$

since, if $a \leq U$, and $b \leq V$ then $ab \leq UV = x$. Thus we again get (3),

$$\begin{aligned} \sum_{(a,b) \in S} f(a)g(b) &= \sum_{ab \leq x, a \leq U} f(a)g(b) + \sum_{ab \leq x, b \leq V} f(a)g(b) \\ &\quad - \sum_{a \leq U, b \leq V} f(a)g(b). \end{aligned}$$

But however we get here, we continue as

$$\begin{aligned} \sum_{(a,b) \in S} f(a)g(b) &= \sum_{a \leq U} f(a) \sum_{b \leq x/a} g(b) + \sum_{b \leq V} g(b) \sum_{a \leq x/b} f(a) \\ &\quad - \sum_{a \leq U} f(a) \sum_{b \leq V} g(b) \\ &= \sum_{a \leq U} f(a) G\left(\frac{x}{a}\right) + \sum_{b \leq V} g(b) F\left(\frac{x}{b}\right) - F(U)G(V). \end{aligned}$$

■

The Hyperbolic Method can be used to improve (2) to

Theorem 3 *For the divisor function we have*

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(x^{1/2}).$$

Proof The hyperbola method gives

$$\sum_{n \leq x} d(n) = \sum_{n \leq x} 1 * 1(n) = \sum_{a \leq U} \sum_{b \leq x/a} 1 + \sum_{b \leq V} \sum_{a \leq x/b} 1 - \left(\sum_{a \leq U} 1 \right) \left(\sum_{b \leq V} 1 \right), \quad (4)$$

for some $UV = x$ to be chosen. The first term equals

$$\sum_{a \leq U} \left[\frac{x}{a} \right] = \sum_{a \leq U} \left(\frac{x}{a} + O(1) \right) = x \sum_{a \leq U} \frac{1}{a} + O(U).$$

The second term similarly equals

$$\sum_{b \leq V} \left[\frac{x}{b} \right] = \sum_{b \leq V} \left(\frac{x}{b} + O(1) \right) = x \sum_{b \leq V} \frac{1}{b} + O(V).$$

So we have two error terms $O(U)$ and $O(V)$. We now choose U and V to equalise these errors, i.e. choose $U = V = x^{1/2}$. In that case the first two terms in (4) are equal and thus the right hand side of (4) is

$$\begin{aligned} 2 \sum_{a \leq x^{1/2}} \left[\frac{x}{a} \right] - [x^{1/2}]^2 &= 2 \sum_{a \leq x^{1/2}} \left(\frac{x}{a} + O(1) \right) - (x^{1/2} + O(1))^2 \\ &= 2x \sum_{a \leq x^{1/2}} \frac{1}{a} + O \left(\sum_{a \leq x^{1/2}} 1 \right) - (x + O(x^{1/2})) \\ &= 2x \left(\log(x^{1/2}) + \gamma + O \left(\frac{1}{x^{1/2}} \right) \right) + O(x^{1/2}) \\ &\quad - (x + O(x^{1/2})), \end{aligned}$$

having used Theorem 1, which gives the stated result. ■

To improve our result on $\sum_{n \leq x} d_3(n)$ using this improved result for $\sum_{n \leq x} d(n)$ requires a further application of the Hyperbolic method and is left as an exercise for the interested student.

Problems on Hyperbolic Method

The simplest application of the Hyperbolic function is to the divisor function. All the other applications are far more involved, though more interesting in that we need choices of U and V different from $U = V = x^{1/2}$.

1) Use the Hyperbolic method to prove

$$\sum_{n \leq x} 2^{\omega(n)} = \frac{x}{\zeta(2)} \log x + c_1 x + O(x^{2/3}),$$

for some constant c_1 .

Hint Recall a result on Problem Sheet 5, that there exists a constant C_Q say, for which

$$\sum_{b \leq V} \frac{Q_2(b)}{b} = \frac{1}{\zeta(2)} \log V + C_Q + O\left(\frac{1}{V^{1/2}}\right).$$

This is shown in exactly the same way as

$$\sum_{a \leq U} \frac{1}{a} = \log U + \gamma + O\left(\frac{1}{U}\right),$$

which will also have to be used.

2) Let $g(n) = d(n^2)$ where d is the divisor function. In Problem Sheet 2 we have $g = 1 * 1 * 1 * \mu_2 = d * Q_2$. Can the Hyperbolic Method be used to improve previous results on $\sum_{n \leq x} d(n^2)$?

To do so you need to prove

i) There exists a constant C_d say for which

$$\sum_{a \leq U} \frac{d(a)}{a} = \frac{1}{2} \log^2 U + 2\gamma \log U + C_d + O\left(\frac{1}{U^{1/2}}\right).$$

ii) There exists a constant C_Q say, for which

$$\sum_{b \leq V} \frac{Q_2(b)}{b} = \frac{1}{\zeta(2)} \log V + C_Q + O\left(\frac{1}{V^{1/2}}\right).$$

iii) Use the above results in the Hyperbolic Method to show that

$$\sum_{n \leq x} d(n^2) = \frac{1}{2\zeta(2)} x \log^2 x + c_1 x \log x + c_2 x + O(x^{3/4} \log x),$$

for some constants c_1 and c_2 .

3) (Hard) For $k \geq 2$ prove using the Hyperbolic Method and induction that

$$\sum_{n \leq x} d_k(n) = x P_{k-1}(\log x) + O(x^{1-1/k} \log^{k-2} x),$$

where $P_d(y)$ is a polynomial of degree d in y .